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Out-arc pancyclicity of vertices in tournaments☆

Qiaoping Guo [a,b], Shengjia Li [b,*], Yubao Guo [c], Hongwei Li [a,b]

[a] School of Mathematical Sciences, Shanxi University, Taiyuan, 030006, China

[b] Institute of Mathematics and Applied Mathematics, Shanxi University, Taiyuan, 030006, China

[c] Lehrstuhl C für Mathematik, RWTH Aachen, Templergraben 55, 52062 Aachen, Germany

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ABSTRACT

Yao, Guo and Zhang [T. Yao, Y. Guo, K. Zhang, Pancyclic out-arcs of a vertex in a tournament, Discrete Appl. Math. 99 (2000) 245–249.] proved that every strong tournament contains a vertex u such that every out-arc of u is pancyclic. In this paper, we prove that every strong tournament with minimum out-degree at least two contains two such vertices. Yeo [A. Yeo, The number of pancyclic arcs in a k-strong tournament, J. Graph Theory 50 (2005) 212–219.] conjectured that every 2-strong tournament has three distinct vertices $\{x, y, z\}$, such that every arc out of x, y and z is pancyclic. In this paper, we also prove that Yeo's conjecture is true.

1. Introduction

A *tournament* is an orientation of the edges of a complete graph. An l-cycle is a cycle of length l. An arc or a vertex is said to be k-*pancyclic* in a digraph D if it belongs to an l-cycle for all $k \leq l \leq |V(D)|$. For $k = 3$, we also say that the arc or the vertex is *pancyclic*. An arc leaving from a vertex x in a digraph is called an *out-arc* of x.

Let D be a digraph with vertex-set $V(D)$ and arc-set $A(D)$. D is said to be *strong* if, for all $x, y \in V(D)$, there is a path from x to y in D. A directed path from x to y in D is denoted by (x, y)-path. D is called k-*strong* if $|V(D)| \geq k + 1$ and $D - X$ is strong for any set $X \subseteq V(D)$ with $|X| < k$. If D is k-strong, but not $(k + 1)$-strong, then $\sigma(D) = k$ is defined as the *strong connectivity* of D.

In 1980, Thomassen [5] proved that every strong tournament contains a vertex x such that each out-arc of x is contained in a Hamiltonian cycle. In 2000, Yao, Guo and Zhang extended the result of Thomassen and proved that

Theorem 1.1 (*Yao, Guo and Zhang [6]*). *Every strong tournament T has a vertex u such that every out-arc of u is pancyclic.*

For strong tournaments with minimum out-degree one, Yao et al. [6] found an infinite class of strong tournaments, such that each tournament contains exactly one vertex such that its out-arc is pancyclic. The problem arises naturally whether a strong tournament with minimum out-degree two contains two distinct vertices such that all out-arcs of them are pancyclic.

For 3-strong tournaments, we have the following.

Theorem 1.2 (*Yeo [7]*). *Every 3-strong tournament contains two distinct vertices x and y, such that all arcs out of x and all arcs out of y are pancyclic.*

Furthermore, x and y can be chosen such that $x \to y$ and $d^+(x) \leq d^+(y)$.

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* Corresponding author.
E-mail address: shjiali@sxu.edu.cn (S. Li).

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In 2008, Li et al. proved that Yeo's result is also true for 2-strong tournaments.

Theorem 1.3 (*Li, Li and Feng [4]*). *Each strong tournament T with $\sigma(T) = 2$ contains at least 2 vertices v_1, v_2 such that all out-arcs of v_i are pancyclic for $i = 1, 2$.*

In this paper we show that Yeo's result still holds for strong tournaments with minimum out-degree two. In other words, we provide a positive answer to the above problem.

In 2005, Yeo posed the following conjecture for 2-strong tournaments.

Conjecture 1.4 (*Yeo [7]*). *If T is a 2-strong tournament, then T has three distinct vertices, $\{x, y, z\}$, such that every arc out of x, y and z is pancyclic.*

The conjecture cannot be extended much further due to the following theorem.

Theorem 1.5 (*Yeo [7]*). *Let $k \geq 1$ be arbitrary. There exists an infinite class of k-strong tournaments, such that each tournament contains at most 3 vertices, with the property that all arcs out of them are pancyclic.*

Recently, Feng proved that the result of Conjecture 1.4 is true for 3-strong tournaments.

Theorem 1.6 (*Feng [3]*). *Each 3-strong tournament T contains three distinct vertices, all of whose out-arcs are pancyclic.*

In this paper we prove that Yeo's conjecture is true.

2. Preliminaries

This paper will generally follow the notation and terminology defined in [1].

Let $D = (V(D), A(D))$ be a digraph without multiple arcs and loops. We denote the number of vertices of D by $|V(D)|$. A subdigraph induced by a subset $A \subseteq V(D)$ is denoted by $D[A]$. We also write $D - A$ for $D[V(D) - A]$.

If uv is an arc in D, then we say that u *dominates* v and write $u \to v$. We also say that uv is an *out-arc* of u. For disjoint subsets X and Y of $V(D)$, if every vertex of X dominates every vertex of Y, we say X dominates Y and write $X \to Y$. $X \Rightarrow Y$ means that there is no arc from a vertex in Y to a vertex in X. For a tournament, $X \to Y$ if and only if $X \Rightarrow Y$.

For a vertex x in D, the set of all vertices dominating x (dominated by x, respectively) is denoted by $N_D^-(x)$ ($N_D^+(x)$, respectively). Furthermore, $d_D^+(x) = |N_D^+(x)|$ and $d_D^-(x) = |N_D^-(x)|$ are the *out-degree* and the *in-degree* of x in D, respectively. We use $\delta^+(D) = \min\{d_D^+(x) : x \in V(D)\}$ to stand for the *minimum out-degree* of D. When there is no confusion possible, we use $N^+(x),\ N^-(x),\ d^+(x)$ and $d^-(x)$ instead of $N_D^+(x),\ N_D^-(x),\ d_D^+(x)$ and $d_D^-(x)$, respectively.

For a strong digraph $D = (V(D), A(D))$, a set $S \subseteq V(D)$ is called a *separating set* of D if $D - S$ is not strong. Furthermore, S is called a *minimum separating set* of D if $D - S$ is not strong and $\sigma(D) = |S|$. A vertex $u \in V(D)$ is called a *cut vertex* of D if $D - \{u\}$ is not strong. A *strong component* of a digraph D is a maximal induced subdigraph of D which is strong. If D is strong, then D is the only strong component. If D is not strong, then we can partition D into strong components $T_1, T_2, \ldots, T_r$ ($r \geq 2$), such that $T_1 \Rightarrow T_2 \Rightarrow \cdots \Rightarrow T_r$.

Let D be a digraph, let x and y be distinct vertices of D and let P be a (x, y)-path in D. We say that D' is a digraph obtained from D by *contracting* P into a new vertex w, not contained in D, if the following holds: $V(D') = (V(D) \backslash V(P)) \cup \{w\}$, $N_{D'}^+(w) = N_D^+(y) \cap (V(D) \setminus V(P))$, $N_{D'}^-(w) = N_D^-(x) \cap (V(D) \setminus V(P))$ and an arc with both end-points in $V(D) \setminus V(P)$ belongs to D' if and only if it belongs to D. It is easy to see that uwv is a path in D' if and only if uPv is a path in D. Analogously, there exists an l-cycle containing w in D' if and only if there is an $(l - 1 + |V(P)|)$-cycle containing P in D.

In the proofs of our main results, the following results are very useful.

Lemma 2.1 (*Yeo [7]*). *Let D be a k-strong digraph, with $k \geq 1$, and let S be a separating set in D, such that $T = D - S$ is a tournament. Let $T_1, T_2, \ldots, T_r$ ($r \geq 2$) be the strong components of T, such that $T_1 \Rightarrow T_2 \Rightarrow \cdots \Rightarrow T_r$. Then the following holds.*

(i) *At least k vertices in S dominate some vertices in T_1, and at least k vertices in S are dominated by some vertices in T_r.*
(ii) *For every $1 \leq l \leq |V(T)| - 1$, $u \in T_1$ and $v \in T_r$, there exists a (u, v)-path of length l in T.*
(iii) *If $S = \{x\}$, then x is pancyclic in D.*

Lemma 2.2 (*Yeo [7]*). *Let D be a strong digraph, containing a vertex x, such that $D - x$ is a tournament and $d_D^+(x) + d_D^-(x) \geq |V(D)|$. Then there is an l-cycle containing x in D, for all $2 \leq l \leq |V(D)|$.*

Lemma 2.3 (*Yeo [7]*). *Let T be a 2-strong tournament, containing an arc $e = xy$, such that $d^+(y) \geq d^+(x)$. Then e is pancyclic in T.*

Theorem 2.4 (*Camion [2]*). *A non-trivial tournament is strong if and only if it has a Hamiltonian cycle.*

Lemma 2.5. *Let T be a 2-strong tournament containing an arc xy, such that $T - \{x, y\}$ is not strong. Then xy is 4-pancyclic in T.*

Proof. Let D_{xy} be the digraph obtained from T by contracting the arc xy into a vertex z_{xy}. Since T is 2-strong, it is clear that D_{xy} is strong. Since $D_{xy} - z_{xy} = T - \{x, y\}$ is not strong, by Lemma 2.1, part (iii), xy is 4-pancyclic in T. □

3. Main results

Theorem 3.1. *Let T be a strong tournament on n vertices with $\sigma(T) = 1$ and minimum out-degree at least two. Then T contains at least two distinct vertices such that all out-arcs of them are pancyclic.*

Proof. Let v be a cut vertex of T and $T_1, T_2, \ldots, T_t$ $(t \geq 2)$ be the strong components of $T - v$, such that $T_1 \Rightarrow T_2 \Rightarrow \cdots \Rightarrow T_t$. By Theorem 2.4, T_i is a single vertex or contains a Hamiltonian cycle for all $1 \leq i \leq t$. Recall that $d^+(u) \geq 2$ for all $u \in V(T)$. We have that T_t contains at least three vertices.

Assume, without loss of generality, that the vertex v has been chosen such that the terminal strong component of $T - v$ is minimum. This implies that any vertex in T_t is not a cut vertex of T.

Let $x \neq v$ be a vertex such that $d^+(x) = \min\{d^+(u) : u \in V(T) \setminus \{v\}\}$. Then it is easy to see that $x \in V(T_t)$. We first show that x is a desired vertex.

Let $x_1 \in N^+(x)$ be arbitrary. Then $x_1 \in V(T_t) \cup \{v\}$. It is easy to check that if $x \to v$ then xv is pancyclic. We consider the case when $x_1 \in V(T_t)$. Let D be the digraph obtained from T by contracting the arc xx_1 into a vertex w. We first prove that D is strong. Let $u \in V(D - w)$ be arbitrary. It is sufficient to show that D contains a path from u to w and a path from w to u. By the definition of path-contracting, we only need to verify that $T - \{x\}$ contains a (x_1, u)-path and $T - \{x_1\}$ contains a (u, x)-path. Recall that x and x_1 are two vertices in T_t and any vertex in T_t is not a cut vertex of T. We have that $T - \{x\}$ and $T - \{x_1\}$ are strong. Therefore, $T - \{x\}$ contains a (x_1, u)-path and $T - \{x_1\}$ contains a (u, x)-path.

For the strong digraph D, we get

$$\begin{aligned} d_D^+(w) + d_D^-(w) &= d_T^+(x_1) + d_T^-(x) = d_T^+(x_1) + (n - 1 - d_T^+(x)) \\ &= n - 1 + (d_T^+(x_1) - d_T^+(x)) \geq n - 1 = |V(D)|. \end{aligned}$$

By Lemma 2.2, the arc xx_1 is pancyclic in T and x is a desired vertex.

If there is a vertex u in $T - \{v, x\}$ with $d^+(u) = d^+(x)$, then u is also a desired vertex by a similar argument as above. Thus, u and x are two desired vertices and we are done. So assume that

$$d^+(u) > d^+(x) \quad \text{for all } u \in T - \{v, x\}. \tag{1}$$

Let $y \neq v$ be a vertex with minimum out-degree among all vertices in $N^+(x)$. Then $y \in V(T_t)$. We will prove that y or another vertex apart from x, y is the second desired vertex.

Let $y_1 \in N^+(y)$ be arbitrary. Then $y_1 \in V(T_t) \cup \{v\}$, and it is easy to see that the arc yv is pancyclic in T if yv is an arc of T. So we consider the case when $y_1 \in V(T_t)$. Let D_1 be the digraph obtained from T by contracting the arc yy_1 into a vertex w_1. Similarly, recall that any vertex in T_t is not a cut vertex of T. We have that $T - \{y\}$ and $T - \{y_1\}$ are strong. Therefore D_1 is strong.

If $d^+(y_1) \geq d^+(y)$, we get

$$\begin{aligned} d_{D_1}^+(w_1) + d_{D_1}^-(w_1) &= d_T^+(y_1) + d_T^-(y) = d_T^+(y_1) + (n - 1 - d_T^+(y)) \\ &= n - 1 + (d_T^+(y_1) - d_T^+(y)) \geq n - 1 = |V(D_1)|. \end{aligned}$$

By Lemma 2.2, yy_1 is pancyclic in T.

So assume that $d^+(y_1) < d^+(y)$. If $D_1 - w_1 = T - \{y, y_1\}$ is not strong then, since D_1 is strong, Lemma 2.1(iii) implies that w_1 is in i-cycles in D_1 for all $3 \leq i \leq |V(D_1)|$. Therefore, yy_1 is in l-cycles in T for all $4 \leq l \leq n$. By the choice of y, we see that $y_1 \to x$ and yy_1xy is a 3-cycle containing yy_1. Thus, yy_1 is pancyclic in T.

In the following, we assume that $d^+(y_1) < d^+(y)$ and $D_1 - w_1 = T - \{y, y_1\}$ is strong. Let D_2 be the digraph obtained from T by contracting the path xyy_1 into a vertex w_2. By (1), $d^+(y_1) > d^+(x)$ holds. We get

$$\begin{aligned} d_{D_2}^+(w_2) + d_{D_2}^-(w_2) &= d_T^+(y_1) - 1 + d_T^-(x) - 1 = d_T^+(y_1) - 1 + (n - 1 - d_T^+(x)) - 1 \\ &= n - 2 + (d_T^+(y_1) - d_T^+(x) - 1) \geq n - 2 = |V(D_2)|. \end{aligned}$$

If D_2 is strong, then yy_1 is pancyclic in T by Lemma 2.2.

So assume that D_2 is not strong. Since $d_{D_2}^+(w_2) + d_{D_2}^-(w_2) \geq |V(D_2)|$, it is easy to see that w_2 is in a 2-cycle of D_2. This implies that $T - \{x, y, y_1\} = D_2 - w_2$ is not strong. Let $T'_1, T'_2, \ldots, T'_r$ $(r \geq 2)$ be the strong components of $T - \{x, y, y_1\}$, such that $T'_1 \Rightarrow T'_2 \Rightarrow \cdots \Rightarrow T'_r$. It is easy to see that $v \in T'_1$. Since D_2 is not strong, we have that $T'_1 \Rightarrow y_1$ or $x \Rightarrow T'_r$ (or both). Recall that $T - \{y, y_1\}$ is strong. There exists an arc from x to T'_1 and an arc from T'_r to x. Therefore, we have $T'_1 \Rightarrow y_1$.

If T'_r has an arc into y, let D_{yy_1x} be the digraph obtained from T by contracting the path yy_1x into a vertex w_{yy_1x}. Clearly, D_{yy_1x} is strong, but $D_{yy_1x} - w_{yy_1x} = T - \{x, y, y_1\}$ is not strong, which by Lemma 2.1(iii), gives us l-cycles containing yy_1 in T for all $5 \leq l \leq |V(T)|$. Since $d^+(y_1) > d^+(x)$, there exists a vertex w such that $y_1 \to w \to x$, and yy_1wxy is a 4-cycle containing yy_1. As we have already found the 3-cycle yy_1xy containing yy_1, we have that yy_1 is pancyclic in T.

Suppose now that T'_r has no arc into y, i.e., $y \Rightarrow T'_r$. If T'_r contains exactly one vertex x_1, recalling that $v \in T'_1$, then $x_1 \neq v$. Now, $d^+(x_1) \leq 2 \leq d^+(x)$, a contradiction to (1). Therefore, $|V(T'_r)| \geq 3$. Let $l = |V(T'_r)|$ and let $C_r = x_1x_2 \cdots x_lx_1$ be a Hamiltonian cycle of T'_r. First, we show that $|N^-(x) \cap V(T'_r)| \geq 3$. Suppose to the contrary that $|N^-(x) \cap V(T'_r)| \leq 2$. Then we have $|N^+(x) \cap V(T'_r)| \geq l - 2$. Recalling that $x \to y$ and x has at least one arc into T'_1, it holds that $d^+(x) \geq l$. For the vertex, say x_i, with minimum out-degree among all vertices in T'_r, it is clear that $|N^+(x_i) \cap V(T'_r)| \leq \lfloor \frac{l-1}{2} \rfloor$. Thus, we have $d^+(x_i) \leq d^+_{T'_r}(x_i) + 2 \leq \lfloor \frac{l-1}{2} \rfloor + 2 \leq \lfloor \frac{l-1}{2} \rfloor + \lceil \frac{l-1}{2} \rceil + 1 = l - 1 + 1 = l \leq d^+(x)$, a contradiction to (1). Therefore, it holds that

$$|N^-(x) \cap V(T'_r)| \geq 3. \tag{2}$$

By Theorem 1.1, the subtournament T'_r has a vertex, say z, whose out-arcs are pancyclic in T'_r. We will show that z is a desired vertex.

Assume, without loss of generality, that the vertices of the Hamiltonian cycle C_r have been labelled with $x_l = z$. Let $z_1 \in N^+(z)$ be arbitrary, then $z_1 \in V(T'_r) \cup \{x, y_1\}$.

Case 1. $z_1 \in V(T'_r)$.

By the choice of z, the arc zz_1 is in an i-cycle for $i = 3, 4, \ldots, l$. Assume, without loss of generality, that C_r is a Hamiltonian cycle of T'_r which contains the arc zz_1, i.e., $z_1 = x_1$.

Recalling that $T - \{y, y_1\}$ is strong, there exists an arc from x to T'_1. Since x is not a cut vertex of T, we have that y has at least one arc into T'_1. Let $u_1u_2 \cdots u_{n-3-l}$ ($u'_1u'_2 \cdots u'_{n-3-l}$, respectively) be a Hamiltonian path of $T - (V(T'_r) \cup \{x, y, y_1\})$ with $x \to u_1$ ($y \to u'_1$, respectively). Note from (2) that there is an integer $k \in \{1, 2, \ldots, l-2\}$ such that $x_k \to x$. It is easy to see that $zx_1 \cdots x_kxu_1x_{k+2} \cdots x_l$ is a cycle of length $l + 1$ and $zx_1 \cdots x_kxu_1 \cdots u_jx_{k+1} \cdots x_l$ is a cycle of length $l + 1 + j$ for $j = 1, 2, \ldots, n - 3 - l$ and $zx_1 \cdots x_kxyu'_1 \cdots u'_{n-3-l}x_{k+1} \cdots x_l$ is a cycle of length $n - 1$. It remains to verify that zx_1 is in a Hamiltonian cycle of T.

Since $T - \{x\}$ is strong, $T - \{x\}$ contains a path from y_1 to T'_1. Recall that $T'_1 \Rightarrow y_1$ and T'_r has no arc into y. We must have $r > 2$ and there exists $u'_j \in T'_i$ for some $1 < i < r$, $1 < j \leq n - 3 - l$ such that $y_1 \to u'_j$. Without loss of generality, let j be the minimum index with this property. Now, $zx_1 \cdots x_kxyu'_1 \cdots u'_{j-1}y_1u'_j \cdots u'_{n-3-l}x_{k+1} \cdots x_l$ is a Hamiltonian cycle of T containing zx_1.

Case 2. $z_1 \in \{x, y_1\}$.

We first consider the arc zx if $z \to x$. Recalling that $T - (V(T'_r) \cup \{x, y, y_1\})$ contains a Hamiltonian path $u_1u_2 \cdots u_{n-3-l}$ with $x \to u_1$, it is easy to see that zx is pancyclic in $T - \{y, y_1\}$. In addition, using the observations from Case 1 above, $zxyu'_1u'_2 \cdots u'_{n-3-l}x_1 \cdots x_l$ is an $(n-1)$-cycle and $zxyu'_1 \cdots u'_{j-1}y_1u'_j \cdots u'_{n-3-l}x_1 \cdots x_l$ is an n-cycle containing zx. Therefore, zx is pancyclic in T.

If zy_1 is an arc of T, we have that $zy_1u'_jz$ and zy_1xu_1z are 3- and 4-cycles containing zy_1, respectively. Since $y_1 \to x$ and $x \to u_1$, it is easy to see that zy_1 is pancyclic in $T - \{y\}$. Moreover, zy_1 is in a Hamiltonian cycle $zy_1xyu'_1 \cdots u'_{n-3-l}x_1 \cdots x_l$ of T. Therefore, zy_1 is pancyclic in T.

The proof of the theorem is complete. $\square$

Combining Theorems 1.2, 1.3 and 3.1, we get the following.

Corollary 3.2. *Each strong tournament T with $\delta^+(T) \geq 2$ contains at least two distinct vertices such that all out-arcs of them are pancyclic.*

As an application of Corollary 3.2, we prove that Conjecture 1.4 is true. Our proof depends on the proof of Theorem 1.3. Theorem 1.3 showed that each strong tournament T with $\sigma(T) = 2$ contains at least two vertices whose out-arcs are pancyclic. Now, our aim is to find the third vertex whose out-arcs are pancyclic on the basis of Theorem 1.3. For convenience, we summarize the results of Theorem 1.3 in the following Remark 1 (we use u, v_1 instead of v_1, v_2 in Theorem 1.3 here).

Remark 1. Let T be a 2-strong tournament with $\sigma(T) = 2$. Let M be the set of all vertices with minimum out-degree in T.

If $|M| = 2$, then the out-arcs of the vertices in M are pancyclic in T.

If $|M| = 1$ and u is the vertex with minimum out-degree in T, then the one of the following two cases holds.

Case 1. $\sigma(T - u) = 1$, $S = \{u, x\}$ is a minimum separating set of T and $T_1, T_2, \ldots, T_t$ ($t \geq 2$) are strong components of $T - S$ such that $T_1 \Rightarrow T_2 \Rightarrow \cdots \Rightarrow T_t$. Assume, without loss of generality, that among all possible minimum separating sets $\{u, x'\}$ with $x' \in V(T - u)$, the vertex $x \in S$ has been chosen such that the terminal strong component of $T - S$ is minimum. Then u and another vertex are two vertices such that all out-arcs of them are pancyclic in T. That is,

(i) If $t = 2$, $T_1 = \{w_1\}$ is a single vertex and $N^+(u) = \{x, w_1\}$, then u and w_1 are two vertices whose out-arcs are pancyclic in T.

(ii) Otherwise, u and a vertex whose out-arcs are pancyclic in T_t are two vertices whose out-arcs are pancyclic in T.

Case 2. $\sigma(T - u) = 2$ and $d^+(v_1) = \min\{d^+(x) : x \in N^+(u)\}$, then u and v_1 are two vertices whose out-arcs are pancyclic in T.

By repeating the proof of Case 1 in Theorem 1.3, we also obtain the following.

Remark 2. In Case 1 in Remark 1, if $d^+(x) = d^+(u)$, then (i) and (ii) in Case 1 above still hold.

Theorem 3.3. *Each strong tournament T on n vertices with $\sigma(T) = 2$ contains at least three distinct vertices whose out-arcs are pancyclic.*

Proof. Let $M = \{x : d^+(x) = \delta^+(T)\}$. By Lemma 2.3, all out-arcs of every vertex in M are pancyclic. If $|M| \geq 3$, then we are done. So assume that $|M| \leq 2$.

In the following, we choose u and v_1 satisfying the following **conditions**.

(1) $d^+(u) = \delta^+(T)$.

(2) $u \to v_1$ and $d^+(v_1) = \min\{d^+(x) : x \in N^+(u)\}$.

(3) When $|M| = 2$, we always assume that $d^+(u) = d^+(v_1) = \delta^+(T)$.

We consider the following two cases.

Case 1. $\sigma(T - u) = 1$.

Let $S = \{u, x\}$ be a minimum separating set of T and let $T_1, T_2, \ldots, T_r$ $(r \geq 2)$ be the strong components of $T - S$, such that $T_1 \Rightarrow T_2 \Rightarrow \cdots \Rightarrow T_r$. Then $|V(T_r)| \geq 3$ by the conditions (2) and (3). Otherwise, T_r only contains one vertex, say u_0. Now, $d^+(u_0) = d^+(u) = 2$ and $u_0 \to u$. This implies that $|M| = 2$ and $u_0 = v_1$, which is a contradiction to $u \to v_1$. Assume, without loss of generality, that among all possible minimum separating sets $\{u, x'\}$ with $x' \in V(T - u)$, the vertex $x \in S$ has been chosen such that the terminal strong component of $T - S$ is minimum. This implies that

$$|N^-(x) \cap V(T_r)| \geq 2. \tag{3}$$

Subcase 1.1. $|M| = 1$.

By Remark 1, we have the following (i) and (ii).

(i) If $t = 2$, $T_1 = \{w_1\}$ is a single vertex and $N^+(u) = \{x, w_1\}$, then u and w_1 are two vertices whose out-arcs are pancyclic in T.

(ii) Otherwise, the vertex u and a vertex whose out-arcs are pancyclic in T_r are two vertices whose out-arcs are pancyclic in T.

In the following, we will find the third desired vertex.

For the case (i), we consider the vertex, say x_0, with minimum out-degree among all vertices in T_2. We will show that x or x_0 is a desired vertex.

If $d^+(x) \leq d^+(x_0)$, let $y \in N^+(x)$ be arbitrary. By the choice of x_0, we have $d^+(y) \geq d^+(x_0) \geq d^+(x)$. By Lemma 2.3, xy is pancyclic. Now, x is a vertex whose out-arcs are pancyclic in T.

If $d^+(x) > d^+(x_0)$, let $z \in N^+(x_0)$ be arbitrary. Then $z \in V(T_2) \cup \{u, x\}$. By the choice of x_0, we have $d^+(z) \geq d^+(x_0)$ for all $z \in V(T_2) \cup \{x\}$. By Lemma 2.3, x_0z is pancyclic. In addition, if $x_0 \to u$, it is easy to check that x_0u is pancyclic. Thus, x_0 is a vertex whose out-arcs are pancyclic in T.

For the case (ii), if T_r contains at least two vertices whose out-arcs are pancyclic in T_r, then we are done.

So we assume that T_r only contains one vertex, say v, whose out-arcs are pancyclic in T_r. Then v is a desired vertex. By Corollary 3.2, we have that $\sigma(T_r) = 1$ and $|N^+(v) \cap V(T_r)| = 1$. Let $N^+(v) \cap V(T_r) = \{w\}$. Recall that $d^+(v) > d^+(u) \geq 2$. We have that $v \to \{u, x\}$ and $N^+(v) = \{u, x, w\}$. So we get

$$d^+(v) = 3 \quad \text{and} \quad d^+(u) = 2. \tag{4}$$

Below, we will show that w or another vertex is a desired vertex.

Let $w_1 \in N^+(w)$ be arbitrary. By Lemma 2.3, we only need to consider the case when $d^+(w_1) < d^+(w)$.

Assume that T_r contains $l \geq 3$ vertices. Thus $d^+(u) < l$. By Theorem 2.4, T_r has a Hamiltonian cycle $x_1x_2 \cdots x_lx_1$ with $x_l = w$. We first show that $|N^-(u) \cap V(T_r)| \geq 2$. Suppose to the contrary that $|N^-(u) \cap V(T_r)| \leq 1$, then $|N^+(u) \cap V(T_r)| \geq l - 1$. By Lemma 2.1, part (i), we have that u dominates at least one vertex in T_1; thus $d^+(u) \geq l$, which is a contradiction. Therefore, we get

$$|N^-(u) \cap V(T_r)| \geq 2. \tag{5}$$

We consider the following two subcases.

Subcase 1.1.1. $w_1 \in \{u, x\}$.

It is obvious (using Lemma 2.1(i)) that the subtournament $T - (\{u, x\} \cup V(T_r))$ contains a Hamiltonian path, say $u_1u_2 \cdots u_{n-2-l}$ ($u'_1u'_2 \cdots u'_{n-2-l}$, respectively), with $u \to u_1$ ($x \to u'_1$, respectively). It is easy to see that wu is pancyclic in $T - \{x\}$ and wx is pancyclic in $T - \{u\}$. We only need to verify that each of wu and wx is contained in a Hamiltonian cycle of T.

For convenience, we denote the other vertex in $\{u, x\}$ by w'_1, i.e., $\{w_1, w'_1\} = \{u, x\}$.

Suppose that $T - (\{w_1, w'_1\} \cup V(T_r))$ contains at least two vertices. Since T is 2-strong, it is not difficult to show that $T - (\{w_1, w'_1\} \cup V(T_r))$ can be decomposed into two paths, say $y_1y_2 \cdots y_k$ and $y'_1y'_2 \cdots y'_m$ with $w_1 \to y_1$ and $w'_1 \to y'_1$, respectively. By (3) and (5), there is a vertex x_j with $1 \leq j \leq l - 1$ such that $x_j \to w'_1$. Now, $ww_1y_1y_2 \cdots y_kx_1x_2 \cdots x_jw'_1y'_1y'_2 \cdots y'_mx_{j+1} \cdots x_l$ is a Hamiltonian cycle containing ww_1.

Suppose now that $T - (\{w_1, w'_1\} \cup V(T_r))$ contains exactly one vertex w'. Note that $r = 2$ and $\{w_1, w'_1\} \to w'$.

If $w_1 \to w'_1$, then we see that $ww_1w'_1w'x_1x_2 \cdots x_l$ is a Hamiltonian cycle of T. So we consider the case when $w'_1 \to w_1$.

Assume that $d^+(w'_1) \geq 3$, which means that $w'_1 \neq u$. Let D be the digraph obtained from T by contracting the path ww_1w' to a vertex z. Since $d^+(w'_1) \geq 3$, we have $N^+(w'_1) \cap V(T_2) \neq \emptyset$. By (3) and (5), $N^-(w'_1) \cap V(T_2 - w) \neq \emptyset$ holds. Therefore, D is strong.

If $|N^+(w) \cap V(T_2)| < l-2$, then $d^+(w') = l = l-2+2 > d^+(w)$. It follows that

$$\begin{aligned} d_D^+(z) + d_D^-(z) &= d_T^+(w') - 1 + d_T^-(w) - 1 = d_T^+(w') - 1 + (n-1-d_T^+(w)) - 1 \\ &= n-2+(d_T^+(w') - d_T^+(w) - 1) \geq n-2 = |V(D)|. \end{aligned}$$

By Lemma 2.2, ww_1 is in a Hamiltonian cycle of T.

If $|N^+(w) \cap V(T_2)| \geq l-2$ then, since $v \to w$, we have that $\{v, w\} \in V(T_2) \setminus N^+(w)$ and $|N^+(w) \cap V(T_2)| = l-2$. Recall that $N^+(v) \cap V(T_2) = \{w\}$. Thus, w is a cut vertex of T_2. Let $T_1', T_2', \ldots, T_t'$ $(t \geq 2)$ be the strong components of $T_2 - w$, such that $T_1' \Rightarrow T_2' \Rightarrow \cdots \Rightarrow T_t'$. It is easy to see that $T_t' = \{v\}$. By Theorem 1.1, T_{t-1}' contains a vertex, say y, such that all out-arcs of y are pancyclic in T_{t-1}'. It is easy to check that y is a vertex whose out-arcs are pancyclic in T_2. Recalling that we have assumed that T_2 only contains one vertex v with this property, this is a contradiction.

Assume now that $d^+(w_1') < 3$. This implies that $d^+(w_1') = 2$ and $w_1' = u$. We see that $T_2 \to w_1'$; this is the case (i), a contradiction.

Subcase 1.1.2. $w_1 \in V(T_r)$.

Recalling that $N^+(v) = \{u, x, w\}$, then $w_1 \to v$ and ww_1 is in the 3-cycle ww_1vw.

By Lemma 2.5, we can assume that $T - \{w, w_1\}$ is strong. If $d^+(w_1) = d^+(v)$, then $d^+(w_1) = 3$ by (4). So $d^+(y) \geq d^+(w_1)$ for all $y \in V(T) \setminus \{u\}$. By Lemma 2.3, w_1y is pancyclic in T for all $y \in N^+(w_1) \setminus \{u\}$. By a similar argument as in Subcase 1.1.1, it is not difficult to prove that w_1u is pancyclic in T. Now, w_1 is a desired vertex.

If $d^+(w_1) > d^+(v)$, let D be the digraph obtained from T by contracting the path vww_1 into a vertex z. We get

$$\begin{aligned} d_D^+(z) + d_D^-(z) &= d_T^+(w_1) - 1 + d_T^-(v) - 1 = d_T^+(w_1) - 1 + (n-1-d_T^+(v)) - 1 \\ &= n-2+(d_T^+(w_1) - d_T^+(v) - 1) \geq n-2 = |V(D)|. \end{aligned}$$

If D is strong, then ww_1 is 4-pancyclic in T by Lemma 2.2. Since ww_1vw is a 3-cycle containing ww_1, we note that ww_1 is pancyclic in T.

If D is not strong, it is easy to see that z is in a 2-cycle of D since $d_D^+(z) + d_D^-(z) \geq |V(D)|$. This implies that $D - z = T - \{v, w, w_1\}$ is not strong. Let $T_1', T_2', \ldots, T_t'$ $(t \geq 2)$ be the strong components of $T - \{v, w, w_1\}$, such that $T_1' \Rightarrow T_2' \Rightarrow \cdots \Rightarrow T_t'$. Then it is easy to see that $u \in T_1'$ or $u \in T_2'$ (when $u \in T_2'$, we have $T_1' = \{x\}$).

Since $T - \{w, w_1\}$ is strong, we have that v has an arc into T_1'. If T_t' has an arc into w, let D_{ww_1v} be the digraph obtained from T by contracting the path ww_1v into a vertex z_{ww_1v}. Clearly, D_{ww_1v} is strong. Since $D_{ww_1v} - z_{ww_1v} = T - \{v, w, w_1\}$ is not strong, Lemma 2.1(iii) implies that ww_1 is 5-pancyclic in T. Since $d^+(w_1) > d^+(v)$, there exists a vertex, say y, such that $w_1 \to y \to v$, and ww_1yvw is a 4-cycle containing ww_1. As we have already found the 3-cycle ww_1vw containing ww_1, we have that ww_1 is pancyclic in T.

Suppose now that T_t' has no arc into w. Note that the component containing u has at least two vertices and has an arc into w. We have $u \notin T_t'$. Assume that T_t' has l vertices. Then $d^+(u) \geq l+1$. For the vertex, say x_0, with minimum out-degree among all vertices in T_t', it is easy to check that $d^+(x_0) \leq (l-1)+2 = l+1 \leq d^+(u)$, a contradiction to $|M| = 1$.

Subcase 1.2. $|M| = 2$.

Recall that $d^+(u) = d^+(v_1) = \delta^+(T)$. It is clear that all out-arcs of u and v_1 are pancyclic in T by Lemma 2.3. In the following, we will find the third desired vertex.

Recalling that $S = \{u, x\}$ is a minimum separating set, then $v_1 \in \{x\} \cup V(T_r)$.

If $v_1 = x$, by Remark 2, a vertex whose out-arcs are pancyclic in T_r or another vertex is a desired vertex and we are done.

Below, we assume that $v_1 \neq x$. It is easy to see that $v_1 \in V(T_r)$. Let $|T_r| = l \geq 3$. We first show that $|N^-(u) \cap V(T_r)| \geq 2$. If $|N^-(u) \cap V(T_r)| \leq 1$, then $|N^+(u) \cap V(T_r)| \geq l-1$. Since u dominates at least one vertex in T_1, we have $d^+(u) \geq l$. For the vertex v_1 in T_r, it is clear that $|N^+(v_1) \cap V(T_r)| \leq l-2$. So $d^+(v_1) \leq l-2+1 = l-1 < d^+(u)$, a contradiction. Therefore, we get

$$|N^-(u) \cap V(T_r)| \geq 2. \tag{6}$$

Let $w \in N^+(v_1) \cap V(T_r)$ be a vertex with $d^+(w) = \min\{d^+(y) : y \in N^+(v_1) \cap V(T_r)\}$. We will show that w or another vertex apart from u and v_1 is a desired vertex.

Let $w_1 \in N^+(w)$ be arbitrary. By Lemma 2.3, we only need to consider the case when $d^+(w_1) < d^+(w)$.

Subcase 1.2.1. $w_1 \in \{u, x\}$.

For convenience, we still denote the other vertex in $\{u, x\}$ by w_1', i.e., $\{w_1, w_1'\} = \{u, x\}$. Let $x_1x_2 \cdots x_lx_1$ be a Hamiltonian cycle of T_r with $x_l = w$.

By a similar argument as in Subcase 1.1.1 (only by using (6) instead of (5)), we only need to verify that ww_1 is contained in a Hamiltonian cycle of T. And we also get that ww_1 is in a Hamiltonian cycle if $T - (\{w_1, w_1'\} \cup V(T_r))$ contains at least two vertices.

Suppose now that $T - (\{w_1, w_1'\} \cup V(T_r))$ contains exactly one vertex w'. Note that $t = 2$ and $\{w_1, w_1'\} \to w'$.

If $w_1 \to w_1'$, then we see that $ww_1w_1'w'x_1x_2 \cdots x_l$ is a Hamiltonian cycle of T. So we consider the case when $w_1' \to w_1$.

If $d^+(w_1') < 3$, this implies that $d^+(w_1') = 2$ and $w_1' = u$, since $x \neq v_1$. We see that $T_2 \to w_1'$, which is a contradiction to $v_1 \in T_2$ and $u \to v_1$.

So assume that $d^+(w_1') \geq 3$. Let D be the digraph obtained from T by contracting the path ww_1w' to a vertex z. It is easy to check that D is strong.

If $d^+(w') > d^+(w)$, we get

$$\begin{aligned} d_D^+(z) + d_D^-(z) &= d_T^+(w') - 1 + d_T^-(w) - 1 = d_T^+(w') - 1 + (n - 1 - d_T^+(w)) - 1 \\ &= n - 2 + (d_T^+(w') - d_T^+(w) - 1) \geq n - 2 = |V(D)|. \end{aligned}$$

By Lemma 2.2, z is in i-cycles in D for all $2 \leq i \leq n-2$, and hence ww_1 is in a Hamiltonian cycle of T.

If $d^+(w') \leq d^+(w)$, then $d^+(w) \geq l = d^+(w')$. Note that $\{v_1, w\} \in V(T_2) \backslash N^+(w)$. We have $d^+_{T_2}(w) \leq l-2$. Since $d^+(w) \leq d^+_{T_2}(w)+2 \leq l-2+2 = l$, we deduce that $d^+(w) = l$ and $d^+_{T_2}(w) = l-2$. Therefore, $w \to \{w_1, w'_1\}$ and $N^-(w) \cap V(T_2) = \{v_1\}$ hold. By the choice of w, it is easy to see that $N^+(v_1) \cap V(T_2) = \{w\}$. Recalling that $\{u, x\} = \{w_1, w'_1\}$, $u \to v_1$ and $d^+(v_1) = d^+(u) = \delta^+(T) \geq 2$, we get that $d^+(v_1) = d^+(u) = 2$ and $v_1 \to x$. Since $d^+(w'_1) \geq 3$, we have that $w'_1 = x$ and $w_1 = u$.

Since $N^+(v_1) \cap V(T_2) = \{w\}$, we see that w is a cut vertex of T_2. Let $T'_1, T'_2, \ldots, T'_t$ $(t \geq 2)$ be the strong components of $T_2 - w$, such that $T'_1 \Rightarrow T'_2 \Rightarrow \cdots \Rightarrow T'_t$. Then $T'_t = \{v_1\}$. By Theorem 1.1, T'_{t-1} contains a vertex, say y, such that all out-arcs of y are pancyclic in T'_{t-1}.

We will show that y is a vertex whose out-arcs are pancyclic in T. Let $y_1 \in N^+(y)$ be arbitrary. Then $y_1 \in V(T'_{t-1}) \cup \{v_1, u, x\}$. By Theorem 2.4, T'_{t-1} is a single vertex or contains a Hamiltonian cycle. Assume, without loss of generality, that T'_{t-1} contains a Hamiltonian cycle, say $x'_1x'_2 \cdots x'_rx'_1$, with $x'_r = y$. Let $u_1u_2 \cdots u_{n-5-r}$ be a Hamiltonian path of $T_2 - (V(T'_{t-1}) \cup \{v_1, w\})$.

We first consider the case when $y_1 \in V(T'_{t-1})$. Recall that y is a vertex whose out-arcs are pancyclic in T'_{t-1}. Assume, without loss of generality, that $x'_1x'_2 \cdots x'_rx'_1$ is a Hamiltonian cycle containing yy_1 in T'_{t-1}, i.e., $y_1 = x'_1$. It is easy to see that yy_1 is pancyclic in T_2. It remains to verify that yy_1 is in a cycle of length from $n-2$ to n. In fact, $yy_1x'_2 \cdots x'_{r-1}v_1xw'u_1u_2 \cdots u_{n-5-r}x'_r$ is an $(n-2)$-cycle containing yy_1. Also, $yy_1x'_2 \cdots x'_{r-1}v_1wxw'u_1u_2 \cdots u_{n-5-r}x'_r$ and $yy_1x'_2 \cdots x'_{r-1}v_1wxuw'u_1u_2 \cdots u_{n-5-r}x'_r$ are $(n-1)$- and n-cycles containing yy_1, respectively. Thus, yy_1 is pancyclic in T.

If yv_1 is an arc, it is easy to see that yv_1 is pancyclic in T_2. It remains to verify that yv_1 is in cycles of length from $n-2$ to n. We see that $yv_1xw'u_1u_2 \cdots u_{n-5-r}x'_1x'_2 \cdots x'_{r-1}x'_r$ and $yv_1wxw'u_1u_2 \cdots u_{n-5-r}x'_1x'_2 \cdots x'_{r-1}x'_r$ are $(n-2)$- and $(n-1)$-cycles containing yv_1, respectively. And $yv_1wxuw'u_1u_2 \cdots u_{n-5-r}x'_1x'_2 \cdots x'_{r-1}x'_r$ is an n-cycle containing yv_1. Therefore, yv_1 is pancyclic in T.

If yx is an arc, it is easy to see that yx is pancyclic in $T - \{v_1, w\}$. In addition, we see that $yxw'u_1u_2 \cdots u_{n-5-r}x'_1x'_2 \cdots x'_{r-1}v_1wx'_r$ is an $(n-1)$-cycle containing yx and $yxuw'u_1u_2 \cdots u_{n-5-r}x'_1x'_2 \cdots x'_{r-1}v_1wx'_r$ is an n-cycle containing yx. Thus, yx is pancyclic in T.

If yu is an arc, it is clear that yu is pancyclic in $T - \{v_1, w, x\}$. It remains to verify that yu is in cycles of length from $n-2$ to n. In fact, $yuv_1wu_1u_2 \cdots u_{n-5-r}x'_1x'_2 \cdots x'_{r-1}x'_r$ and $yuw'v_1wu_1u_2 \cdots u_{n-5-r}x'_1x'_2 \cdots x'_{r-1}x'_r$ are $(n-2)$- and $(n-1)$-cycles containing yu. And $yuv_1wxw'u_1u_2 \cdots u_{n-5-r}x'_1x'_2 \cdots x'_{r-1}x'_r$ is an n-cycle containing yu. Thus, yu is pancyclic in T.

Subcase 1.2.2. $w_1 \in V(T_r)$.

By the choice of w, we have $w_1 \to v_1$. Now, ww_1v_1w is a 3-cycle containing ww_1.

By Lemma 2.5, we can assume that $T - \{w, w_1\}$ is strong. Let D be the digraph obtained from T by contracting the path v_1ww_1 into a vertex z. We get

$$\begin{aligned} d_D^+(z) + d_D^-(z) &= d_T^+(w_1) - 1 + d_T^-(v_1) - 1 = d_T^+(w_1) - 1 + (n - 1 - d_T^+(v_1)) - 1 \\ &= n - 2 + (d_T^+(w_1) - d_T^+(v_1) - 1) \geq n - 2 = |V(D)|. \end{aligned}$$

If D is strong, then ww_1 is 4-pancyclic in T by Lemma 2.2. Thus, ww_1 is pancyclic in T.

If D is not strong, we have that ww_1 is pancyclic in T by a similar argument as in Subcase 1.1.2.

Case 2. $\sigma(T - u) = 2$.

We first give some claims.

Claim A. *u and v_1 are two vertices whose out-arcs are pancyclic in T.*

Proof. Recalling that $M = \{x : d^+(x) = \delta^+(T)\}$, it is clear that all out-arcs of every vertex in M are pancyclic by Lemma 2.3. When $|M| = 2$, since $d^+(v_1) = d^+(u) = \delta^+(T)$, we are done. When $|M| = 1$, by Case 2 in Remark 1, this claim is also true. $\square$

In the following, we only need to find a desired vertex apart from u and v_1.

For convenience, denote $\delta^+(T) = s$. Then $s \geq 2$. Let $N^+(u) = \{v_1, v_2, \ldots, v_s\}$. By Claim A, if $d^+(v_i) = d^+(v_1)$ for some $2 \leq i \leq s$, then v_i is a desired vertex. So we assume that $d^+(v_1) < d^+(v_2) \leq d^+(v_3) \leq \cdots \leq d^+(v_s)$. Let $N^+(v_1) = \{w_1, w_2, \ldots, w_t\}$ and $d^+(w_1) \leq d^+(w_2) \leq \cdots \leq d^+(w_t)$.

Claim B. *If $v_1 \to v_i$ for some $2 \leq i \leq s$, then there exists a vertex v_k $(k \neq 1)$ such that all out-arcs of v_k are pancyclic in T.*

Proof. Let k be the minimum index such that $v_1 \to v_k$, i.e., $k \geq 2$ and $v_i \to v_1$ for all $1 < i \leq k-1$.

Let $x \in N^+(v_k)$ be arbitrary. By Lemma 2.3, we only need to consider the case when $d^+(x) < d^+(v_k)$.

If $x \in \{v_2, \ldots, v_{k-1}\}$, then $v_kxv_1v_k$ is a 3-cycle containing v_kx. If $x \notin \{v_2, \ldots, v_{k-1}\}$, then we have that $x \notin N^+(u)$ and $x \to u$. Now, v_kxuv_k is a 3-cycle containing v_kx. We will prove that v_kx is 4-pancyclic in T.

By Lemma 2.5, we can assume that $T-\{v_k, x\}$ is strong. Let D be the digraph obtained from T by contracting the path uv_kx into a vertex w. Since $\sigma(T-u)=2$, we have that $T-\{u, v_k\}$ is strong. Therefore, we get that D is strong. Note that $x \neq v_1$ ($v_1 \to v_k$ but $v_k \to x$) and $d_T^+(x) > d_T^+(u)$. We get

$$\begin{aligned} d_D^+(w)+d_D^-(w) &\geq d_T^+(x)-1+d_T^-(u)-1 = d_T^+(x)-1+(n-1-d_T^+(u))-1 \\ &= n-2+(d_T^+(x)-d_T^+(u)-1) \geq n-2 = |V(D)|. \end{aligned}$$

By Lemma 2.2, v_kx is 4-pancyclic in T. □

By Claim B, we can assume that $v_i \to v_1$ for all $2 \leq i \leq s$.

Claim C. *If $w_j \to v_i$ for some $1 \leq i \leq s$ and $1 \leq j \leq t$, then there exists a vertex v_k ($k \neq 1$) all of whose out-arcs are pancyclic in T.*

Proof. Let k be the minimum index such that there is some w_j ($1 \leq j \leq t$) satisfying $w_j \to v_k$ but $v_i \to N^+(v_1) = \{w_1, w_2, \ldots, w_t\}$ for all $1 \leq i \leq k-1$.

Let $x \in N^+(v_k)$ be arbitrary. By Lemma 2.3, we only need to consider the case when $d^+(x) < d^+(v_k)$.

If $x \in \{v_1, v_2, \ldots, v_{k-1}\}$, then v_kx is in the 3-cycle $v_kxw_jv_k$. If $x \notin \{v_1, v_2, \ldots, v_{k-1}\}$, then $x \to u$, and hence v_kxuv_k is a 3-cycle containing v_kx. Thus, by Lemma 2.5, we can assume that $T-\{v_k, x\}$ is strong. Let D be the digraph obtained from T by contracting the path uv_kx into a vertex w. Since $\sigma(T-u)=2$, we have that $T-\{u, v_k\}$ is strong. Therefore, we get that D is strong.

If $x \neq v_1$, then $d_T^+(x) > d_T^+(u)$. We get

$$\begin{aligned} d_D^+(w)+d_D^-(w) &\geq d_T^+(x)-1+d_T^-(u)-1 = d_T^+(x)-1+(n-1-d_T^+(u))-1 \\ &= n-2+(d_T^+(x)-d_T^+(u)-1) \geq n-2 = |V(D)|. \end{aligned}$$

By Lemma 2.2, v_kx is 4-pancyclic in T.

If $x = v_1$, then

$$\begin{aligned} d_D^+(w)+d_D^-(w) &= d_T^+(v_1)+d_T^-(u) = d_T^+(v_1)+(n-1-d_T^+(u)) \\ &= n-1+(d_T^+(v_1)-d_T^+(u)) > n-2 = |V(D)|. \end{aligned}$$

By Lemma 2.2, v_kx is 4-pancyclic in T. As the 3-cycle containing v_kx has been found, we see that v_kx is pancyclic in T. □

By Claim C, one can assume that $N^+(u) \to N^+(v_1)$.

In the following, we will show that w_1 or another vertex is a desired vertex. Let $x \in N^+(w_1)$ be arbitrary. We consider the following two subcases.

Subcase 2.1. $x = u$.

It is clear that $w_1uv_1w_1$ and $w_1uv_2v_1w_1$ are 3- and 4-cycles containing w_1u, respectively. We only need to show that w_1u is 5-pancyclic in T.

If there exists v_i for some $1 \leq i \leq s$, such that $T-\{w_1, u, v_i\}$ is not strong, let D be the digraph obtained from T by contracting the path w_1uv_i into a vertex w. Since $\sigma(T-u)=2$, we have that $T-\{w_1, u\}$ and $T-\{u, v_i\}$ are strong. Therefore, D is strong. By Lemma 2.1, part (iii), w_1u is 5-pancyclic in T.

In the following, we always assume that $T-\{w_1, u, v_i\}$ is strong for all $1 \leq i \leq s$.

If $d^+(v_2) \geq d^+(v_1)+2$, let D' be the digraph obtained from T by contracting the path $v_1w_1uv_2$ into a vertex w'. Since $T-\{v_1, w_1, u\}$ and $T-\{w_1, u, v_2\}$ are strong, we have that D' is strong. Moreover, we get

$$\begin{aligned} d_{D'}^+(w')+d_{D'}^-(w') &= d_T^+(v_2)-2+d_T^-(v_1)-2 = d_T^+(v_2)-2+(n-1-d_T^+(v_1))-2 \\ &= n-3+(d_T^+(v_2)-d_T^+(v_1)-2) \geq n-3 = |V(D')|. \end{aligned}$$

By Lemma 2.2, w_1u is in l-cycles in T for all $5 \leq l \leq n$.

Suppose now that $d^+(v_2) \leq d^+(v_1)+1$. Since $d^+(v_1) < d^+(v_2)$, we have $d^+(v_2) = d^+(v_1)+1$. Then $N^+(v_2) = \{v_1\} \cup N^+(v_1)$. If $d^+(u) \geq 3$, then $v_3 \to v_2$ and $\{v_1, v_2\} \cup N^+(v_1) \subseteq N^+(v_3)$. We have $d^+(v_3) \geq d^+(v_1)+2$. Let D'' be the digraph obtained from T by contracting the path $v_1w_1uv_3$ into a vertex w''. By a similar argument as above, w_1u is 5-pancyclic in T.

So assume that $d^+(u)=2$ and $d^+(v_2) = d^+(v_1)+1$. Now, $N^+(u) = \{v_1, v_2\}$, $N^+(v_1) = \{w_1, w_2, \ldots, w_t\}$ and $N^+(v_2) = \{v_1\} \cup \{w_1, w_2, \ldots, w_t\}$ (when $|M|=2$, we have $t=2$).

If $w_1 \to w_i$ for some $2 \leq i \leq t$, let D be the digraph obtained from T by contracting the path $w_1uv_1w_i$ into a vertex w. We first prove that D is strong. Let $y \in V(D-w) = V(T-\{w_1, u, v_1, w_i\})$ be arbitrary. From the definition of path-contracting, it is sufficient to prove that $T-\{w_1, u, v_1\}$ contains a (w_i, y)-path and $T-\{u, v_1, w_i\}$ contains a (y, w_1)-path. Since $T-\{w_1, u, v_i\}$ is strong for all $1 \leq i \leq s$, $T-\{w_1, u, v_1\}$ contains a (w_i, y)-path. It remains to prove that $T-\{u, v_1, w_i\}$ contains a (y, w_1)-path.

If $y \notin N^+(v_1)$, then $y = v_2$ or $y \to v_2$. Since $v_2 \to w_1$, it is clear that there is a (y, w_1)-path in $T-\{u, v_1, w_i\}$ in both cases. Suppose now that $y \in N^+(v_1)$. If $y \to w_1$, then yw_1 is a (y, w_1)-path. If $w_1 \to y$, there exists a vertex z such that $y \to z \to w_1$ since $d^+(y) \geq d^+(w_1)$. Now, yzw_1 is a (y, w_1)-path.

For the strong digraph D, we have

$$\begin{aligned} d_D^+(w)+d_D^-(w) &= d_T^+(w_i)-1+d_T^-(w_1)-1 = d_T^+(w_i)-1+(n-1-d_T^+(w_1))-1 \\ &= n-3+(d_T^+(w_i)-d_T^+(w_1)) \geq n-3 = |V(D)|. \end{aligned}$$

By Lemma 2.2, w_1u is 5-pancyclic in T.

So assume that $w_i \to w_1$ for all $2 \leq i \leq t$. Since $T-\{w_1, u, v_2\}$ is strong, there exists a Hamiltonian cycle $x_1x_2\cdots x_{n-3}x_1$ in $T-\{w_1, u, v_2\}$ by Theorem 2.4. Assume, without loss of generality, that $x_1 = v_1$. Then $w_1uv_2x_1\cdots x_lw_1$ (if $x_l \in \{w_2, \ldots, w_t\}$) or $w_1ux_1\cdots x_lv_2w_1$ (if $x_l \notin \{w_2, \ldots, w_t\}$) is a cycle of length $l+3$ for $l = 2, 3, \ldots, n-3$.

So w_1u is pancyclic in T.

Subcase 2.2. $x \neq u$.

We only need to consider the case when $d^+(x) < d^+(w_1)$ by Lemma 2.3. Thus $x \notin \{w_2, \ldots, w_t\}$ and so $x \to v_1$. It is clear that $x \to u$. So w_1x is in the 3-cycle $w_1xv_1w_1$, the 4-cycle $w_1xuv_1w_1$ and the 5-cycle $w_1xuv_2v_1w_1$.

We first prove the following claim.

Claim D. *If $x \to v_i$ for some $1 \leq i \leq s$ and $T-\{v_i, w_1, x\}$ is not strong, then w_1x is pancyclic in T.*

Proof. Let $D_{w_1xv_i}$ be the digraph obtained from T by contracting the path w_1xv_i into a vertex $w_{w_1xv_i}$. If $D_{w_1xv_i}$ is strong, by Lemma 2.1, part (iii), w_1x is 5-pancyclic in T. As we have found the desired 3- and 4-cycles, we have that w_1x is pancyclic in T.

If $D_{w_1xv_i}$ is not strong, let $T_1, T_2, \ldots, T_r$ ($r \geq 2$) be the strong components of $T-\{w_1, x, v_i\}$, such that $T_1 \Rightarrow T_2 \Rightarrow \cdots \Rightarrow T_r$. As $D_{w_1xv_i}$ is not strong, either $T_1 \Rightarrow v_i$ or $w_1 \Rightarrow T_r$ (or both).

By Lemma 2.5, we can assume that $T-\{w_1, x\}$ is strong. Then v_i has an arc into T_1 and we have $w_1 \Rightarrow T_r$. Since $d^+(u) = \delta^+(T)$, it is easy to see that $u \in V(T_r)$. Since $u \to v_1$ and $u \to v_2$ and $x \notin \{v_1, v_2\}$, we have that v_1 or v_2 must belong to $V(T_r)$. Now, $w_1 \to v_1$ or $w_1 \to v_2$, since $w_1 \Rightarrow T_r$. This is a contradiction to $\{v_1, v_2\} \subseteq N^-(w_1)$. $\square$

In the following, we consider two subcases, (a) and (b).

(a) If $d^+(x) \geq d^+(u)+2$, let D be the digraph obtained from T by contracting the path uv_1w_1x into a vertex w. We have

$$\begin{aligned} d_D^+(w)+d_D^-(w) &= d_T^+(x)-2+d_T^-(u)-2 = d_T^+(x)-2+(n-1-d_T^+(u))-2 \\ &= n-3+(d_T^+(x)-d_T^+(u)-2) \geq n-3 = |V(D)|. \end{aligned}$$

If D is strong, by Lemma 2.2, w_1x is 5-pancyclic in T.

If D is not strong, it is easy to see that w is in a 2-cycle of D, since $d_D^+(w)+d_D^-(w) \geq |V(D)|$, which implies that $D-w = T-\{u, v_1, w_1, x\}$ is not strong. Recalling that $x \to v_1$, we can assume that $T-\{v_1, w_1, x\}$ is strong by Claim D. Since D is not strong, by the definition of path-contracting, we have that $T-\{u, v_1, w_1\}$ is not strong. Let $T_1, T_2, \ldots, T_k$ ($k \geq 2$) be the strong components of $T-\{u, v_1, w_1\}$, such that $T_1 \Rightarrow T_2 \Rightarrow \cdots \Rightarrow T_k$. Since $\sigma(T-u) = 2$, we see that $T-\{u, v_1\}$ and $T-\{u, w_1\}$ are strong. Then each of v_1 and w_1 has an arc into T_1 and T_k has arcs into each of v_1 and w_1. Recalling that $N^+(v_1) \subseteq N^+(v_2)$, then v_2 has at least one arc into T_1. We have $v_2 \in V(T_1)$. Since $u \to v_2$, we have that u has an arc into T_1.

By Theorem 1.1, T_k contains a vertex x_1 whose out-arcs are pancyclic in T_k.

If $T_k = \{x_1\}$ is a single vertex, recalling that $d^+(x_1) > d^+(u) \geq 2$, it is clear that $x_1 \to \{u, v_1, w_1\}$ and it is easy to check that x_1z is pancyclic in T for any $z \in \{u, v_1, w_1\}$.

Assume that $|V(T_k)| \geq 3$. It is not difficult to show that $|N^-(u) \cap V(T_k)| \geq 3$, and it is easy to check that x_1 is a vertex whose out-arcs are pancyclic in T.

(b) If $d^+(x) = d^+(u)+1$, it follows that $d^+(y) \geq d^+(x)$ for all $y \in V(T) \setminus \{u, v_1\}$. By Lemma 2.3, we see that xy is pancyclic for all $y \in N^+(x) \setminus \{u, v_1\}$.

For the arc xv_1, by Claim D, we can assume that $T-\{v_1, w_1, x\}$ is strong. Let D be the digraph obtained from T by contracting the path xv_1w_1 into a vertex w. Since $d^+(w_1) > d^+(x)$, we get

$$\begin{aligned} d_D^+(w)+d_D^-(w) &\geq d_T^+(w_1)-1+d_T^-(x)-1 = d_T^+(w_1)-1+(n-1-d_T^+(x))-1 \\ &= n-2+(d_T^+(w_1)-d_T^+(x)-1) \geq n-2 = |V(D)|. \end{aligned}$$

It is easy to see that w is in a 2-cycle of D. Since $D-w = T-\{v_1, w_1, x\}$ is strong, we have that D is strong. By Lemma 2.2, xv_1 is 4-pancyclic in T. Since xv_1 is in the 3-cycle xv_1w_1x, we have that xv_1 is pancyclic in T.

For the arc xu, let D_1 the digraph obtained from T by contracting the path xuv_1 into a vertex w_1. Since $\sigma(T-u) = 2$, it is easy to see that D is strong. Moreover, we get

$$\begin{aligned} d_{D_1}^+(w_1)+d_{D_1}^-(w_1) &\geq d_T^+(v_1)+d_T^-(x) = d_T^+(v_1)+(n-1-d_T^+(x)) \\ &= n-2+(d_T^+(v_1)+1-d_T^+(x)) \geq n-2 = |V(D_1)|. \end{aligned}$$

By Lemma 2.2, xu is 4-pancyclic in T. If xu is in a 3-cycle of T, then x is a desired vertex. So assume that there is no 3-cycle containing xu. This implies that $x \to N^+(u)$.

By Claim D, we can assume that $T-\{v_i, w_1, x\}$ is strong for all $1 \leq i \leq s$.

If $d^+(v_2) \geq d^+(v_1) + 2$, let D' be the digraph obtained from T by contracting the path $v_1w_1xv_2$ into a vertex w'. We have

$$\begin{aligned} d^+_{D'}(w') + d^-_{D'}(w') &= d^+_T(v_2) - 2 + d^-_T(v_1) - 2 = d^+_T(v_2) - 2 + (n - 1 - d^+_T(v_1)) - 2 \\ &= n - 3 + (d^+_T(v_2) - d^+_T(v_1) - 2) \geq n - 3 = |V(D')|. \end{aligned}$$

Since $T - \{v_1, w_1, x\}$ and $T - \{w_1, x, v_2\}$ are strong, we have that D' is strong. By Lemma 2.2, w_1x is 5-pancyclic in T.

Suppose now that $d^+(v_2) \leq d^+(v_1) + 1$. Since $d^+(v_1) < d^+(v_2)$, we have $d^+(v_2) = d^+(v_1) + 1$. Then $N^+(v_2) = \{v_1\} \cup N^+(v_1)$. If $d^+(u) \geq 3$, then $v_3 \to v_2$ and $\{v_1, v_2\} \cup N^+(v_1) \subseteq N^+(v_3)$. We have $d^+(v_3) \geq d^+(v_1) + 2$. Let D'' be the digraph obtained from T by contracting the path $v_1w_1xv_3$ into a vertex w''. By a similar argument as above, w_1x is 5-pancyclic in T. So assume that $d^+(u) = 2$ and $d^+(v_2) = d^+(v_1) + 1$. Now, $T - \{v_1, w_1, x, v_2\}$ is not strong. Let D''' be the digraph obtained from T by contracting the path $v_1w_1xv_2$ into a vertex w'''. Since $T - \{v_1, w_1, x\}$ and $T - \{w_1, x, v_2\}$ are strong, we have that D''' is strong. Since $D''' - w''' = T - \{v_1, w_1, x, v_2\}$ is not strong, by Lemma 2.1, part (iii), w_1x is 6-pancyclic in T. As we have found the desired 3-, 4- and 5-cycles, we note that w_1x is pancyclic in T.

So the proof of Theorem 3.3 is complete. $\square$

Combining Theorems 1.6 and 3.3, Yeo's conjecture has been proven.

Corollary 3.4. *Each 2-strong tournament T contains at least three distinct vertices whose out-arcs are pancyclic in T.*

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